Quantum Corrected Black Hole Entropy & EFT Transitions

Alberto Castellano Mora

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UW Madison, HEP & Cosmology Seminar

KADANOFF CENTER FOR THEORETICAL PHYSICS
Kavli Institute for Cosmological Physics at the University of Chicago
ENRICO FERMI INSTITUTE
THE UNIVERSITY OF CHICAGO

Based on:

- *Black Hole Entropy, Quantum Corrections and EFT Transitions*, **A. Castellano**, M. Zatti, [arXiV:2502.02655]

- *The Double EFT Expansion in Quantum Gravity* *J. Calderón-Infante*, **A. Castellano**, A. Herráez, [arXiV:2501.14880]

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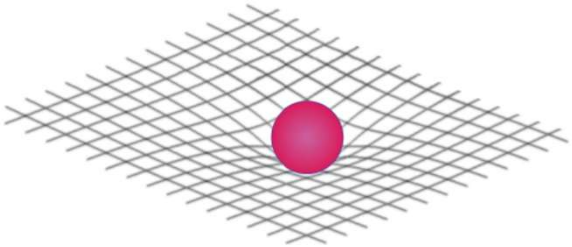
Structure of Grav. EFTs

- At low energies (i.e. $E \ll M_{Pl}$) Gravity is well-described by an Effective Field Theoy (EFT)

$$S_{\rm EH}\left[g_{\mu\nu}\right] = \frac{1}{2\kappa_d^2}\int d^dx\sqrt{-g}\left(\mathcal{R} - 2\Lambda_{\rm c.c.}\right)$$



- Beyond the two-derivative Einstein-Hilbert action one expects further terms [e.g., Donoghue '94]

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- The derivative expansion is controlled by the Quantum Gravity scale $\Lambda_{\rm QG} \overset{?}{\simeq} M_{\rm Pl}$

[v. d. Heisteeg, Vafa, Wiesner, Wu '22-23
AC, Herráez, Ibáñez '23, J. Calderon, AC, A. Herraez '25]

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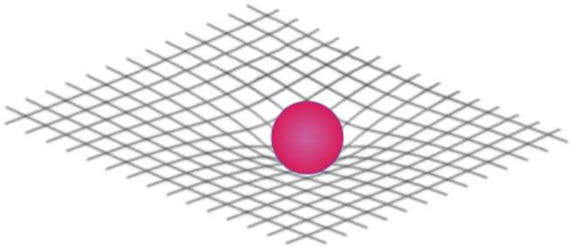

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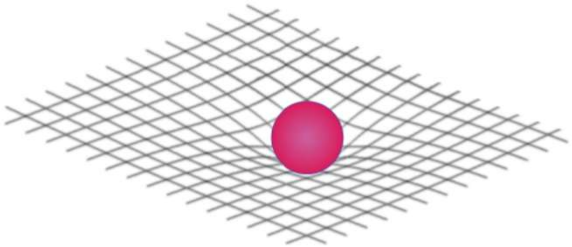

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Higher-curv. ops. are suppressed by QG scale!

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- Top-down evidence (i.e. higher dim., strings, etc.) tells us this is not the end of the story!

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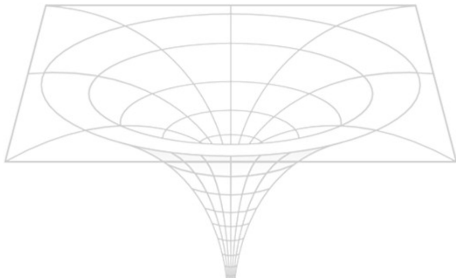
- The scale M is usually associated to (milder) EFT breakdown (mass states, KK modes, etc.)
- This has nice implications for S-matrix bootstrap and gravitational amplitudes [J. Calderon, AC, A. Herraez '25]
- Question 4 today: How do black holes know about these two (very different) scales?



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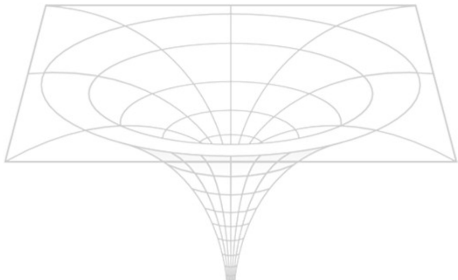

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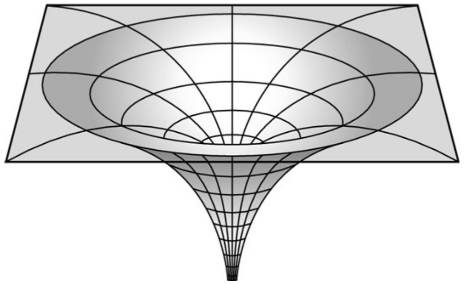

Black Holes & Hidden Scales

- Consider a simple scenario exhibiting two such different scales: Decompact. Limit

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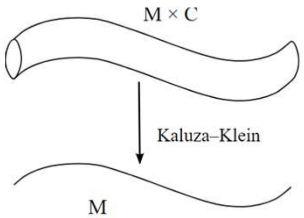

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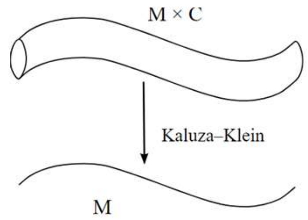

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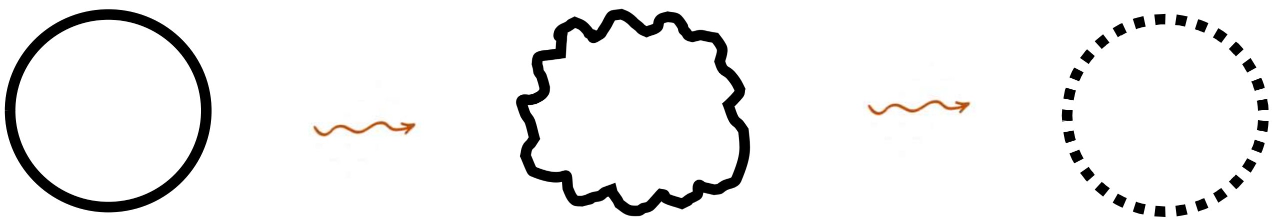
- Question 1: Do **neutral** black holes know about M_{KK}? $\rightsquigarrow$ Answer: Yes! [Gregory, Laflamme '93]



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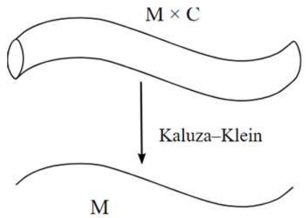

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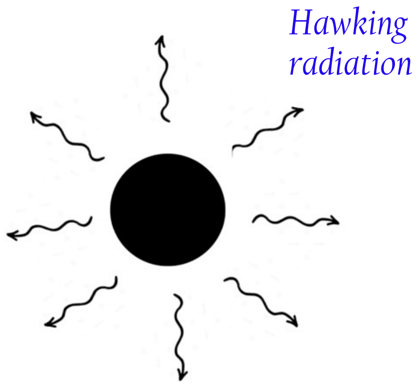
- Question 2: How does QG scale affect black hole sols? $\rightsquigarrow$ Answer: Minimal BH (entropy)!



[Dvali, (Redi) '07]

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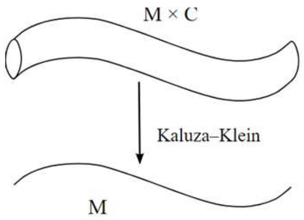

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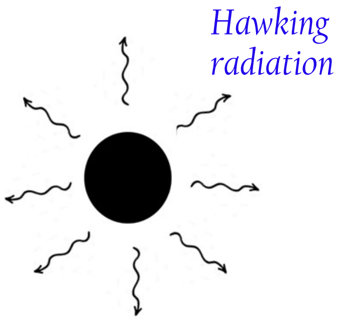

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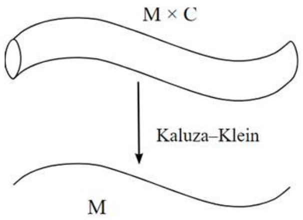

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- $$R_{\mathrm{BH,\,min}} \sim \Lambda_{\mathrm{QG}}^{-1} \rightsquigarrow S_{\mathrm{BH}} \gtrsim \left(\frac{M_{\mathrm{Pl}}}{\Lambda_{\mathrm{QG}}}\right)^{d-2}$$

In This Talk...

- Main goal: Illustrate these expectations in a controlled setup
- We consider supersymmetric theories and BPS objects $\rightsquigarrow$ Extremal BHs
- In particular, we focus on 4d N=2 theories and investigate how these two scales show up in the quantum-corrected BH entropy

1. When $R_{\rm BH} \sim M_{\rm KK}^{-1}$ the low-dim EFT no longer provides a good estimate of BH entropy (EFT transition)

2. The minimal BH entropy is attained for quantum 'effective' areas of $\mathcal{O}(\Lambda_{\rm QG}^{-1})$

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I. Review: 4d N=2 BPS Black Holes

II. Gluing Entropies Across Dimensions

   i. The D0-D2-D4 System

   ii. Perturbative Corrections and Non-Local Resummation

   iii. Leading Non-Perturbative Effects

III. The Fate of Other BPS Systems

   i. The D2-D6 System

   ii. A closer look @ non-perturbative effects

IV. Summary and Outlook

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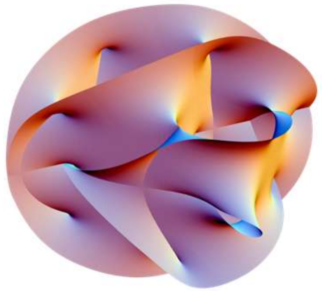
Part I

Review: 4d N=2 BPS Black Holes

4d N=2 Theories: The Lagrangian

- Consider 4d theories preserving 8 supercharges. E.g., take Type IIA on CY 3-fold



- The bosonic action reads (@ 2-derivative level)

$$S_{\rm IIA}^{\rm 4d} = \frac{1}{2\kappa_4^2}\int \mathcal{R}\star 1 + \frac{1}{2}{\rm Re}\,\mathcal{N}_{AB}F^A\wedge F^B + \frac{1}{2}{\rm Im}\,\mathcal{N}_{AB}F^A\wedge\star F^B$$
$$-\frac{1}{\kappa_4^2}\int G_{a\bar b}\,dz^a\wedge\star d\bar z^b + h_{pq}\,dq^p\wedge\star dq^q\,,$$

- The moduli space factorizes between vector and hypermultiplets

$$\mathcal{M}_{\rm mod} = \mathcal{M}_{\rm VM}\times\mathcal{M}_{\rm HM}$$

- In what follows we will restrict to the vector multiplet sector

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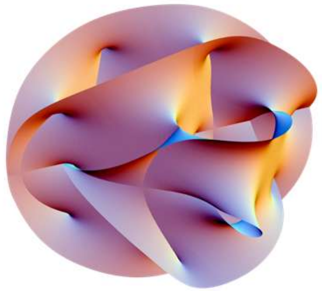

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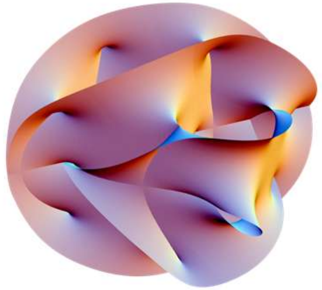

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4d N=2 Theories: The Lagrangian

- The vector multiplet sector is a projective special Kähler manifold

$$G_{a\bar{b}} = \partial_a \partial_{\bar{b}} K, \quad \text{with} \quad K = -\log i \left(\bar{X}^A(\bar{z}) \mathcal{F}_A(z) - X^A(z) \bar{\mathcal{F}}_A(\bar{z}) \right) , \quad z^a = \frac{X^a}{X^0}$$

- The latter is completely determined by the prepotential

$$\mathcal{F} = \frac{1}{2} X^A \mathcal{F}_A, \quad \text{where} \quad \mathcal{F}_A = \partial_{X^A} \mathcal{F}$$

- Moreover, N=2 supersymmetry fixes the gauge kin. matrix in terms of previous quantities

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- Beyond two derivatives, there exist interesting higher-dimensional BPS operators

$$\mathcal{L}_{\text{h.d.}} \supset \sum_{g\geq 1} \int \mathrm{d}^4\theta\, \mathcal{F}_g(\mathcal{X}^A)\ \left(\mathcal{W}^{ij}\mathcal{W}_{ij}\right)^g \ +\ \text{h.c.}$$

[Antoniadis, Gava, Narain, Taylor '95]

- This includes higher-curvature/derivative ops of the form

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- There are further terms linear in Riem and quadratic in W, which are also important

$$W^{ij,-}_{\mu\rho}\nabla^\rho\nabla^\sigma W^{kl,-}_{\sigma\nu}\epsilon_{ik}\epsilon_{jl}\,, \quad \text{with}\ \ W^-_{\mu\nu} = 2ie^{K/2}\text{Im}\,\mathcal{N}_{AB}X^A F^{B,-}_{\mu\nu}\,, \qquad W^{ij,-}_{\mu\nu} = \frac{\epsilon^{ij}}{2}W^-_{\mu\nu}$$

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Scalars *Graviton & graviphoton (anti-self-dual)*

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[Antoniadis, Gava, Narain, Taylor '95]

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$$\mathcal{L}_{\text{h.d.}} \supset \sum_{g\geq 1} \mathcal{F}_g(X^A) \, \mathcal{R}_-^2 \, W_-^{2g-2} \; + \; \text{h.c.}\,,$$

Scalars

Graviton & graviphoton (anti-self-dual)

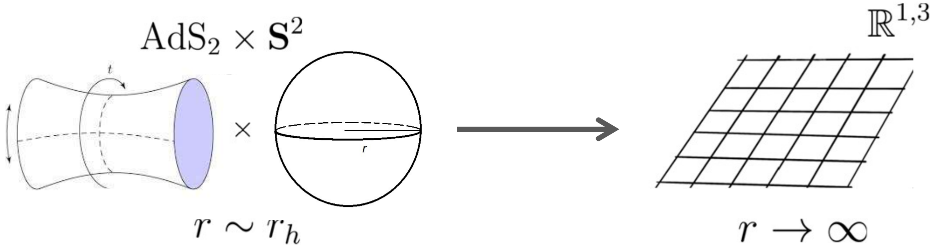
- There are further terms linear in Riem and quadratic in *W*, which are also important

$$W^{ij,-}_{\mu\rho} \nabla^\rho \nabla^\sigma W^{kl,-}_{\sigma\nu} \, \epsilon_{ik}\epsilon_{jl}\,, \quad \text{with} \;\; W^-_{\mu\nu} = 2ie^{K/2} \text{Im}\, \mathcal{N}_{AB} X^A F^{B,-}_{\mu\nu}\,, \qquad W^{ij,-}_{\mu\nu} = \frac{\epsilon^{ij}}{2} W^-_{\mu\nu}$$

4d N=2 Theories: BPS BHs

- An interesting class of objects are BPS (extremal) black holes

$$ds^2 = -e^{2U(r)}dt^2 + e^{-2U(r)}\left(g(r)^{-2}dr^2 + r^2 d\Omega_2^2\right)$$



- Physical properties characterized by gauge charges (attractor mechanism) [Ferrara, Kallosh, Strominger '95]

- This can be generalized to include higher-derivative corrections! [Lopes-Cardoso, Wit, Mohaupt '98-'99]

- We introduce rescaled variables and (symplectic) generalizations thereof

$$Y^A = e^{\mathcal{K}/2}\bar{\mathcal{Z}}X^A \qquad |\mathcal{Z}|^2 = p^A F_A(Y,\Upsilon) - q_A Y^A$$
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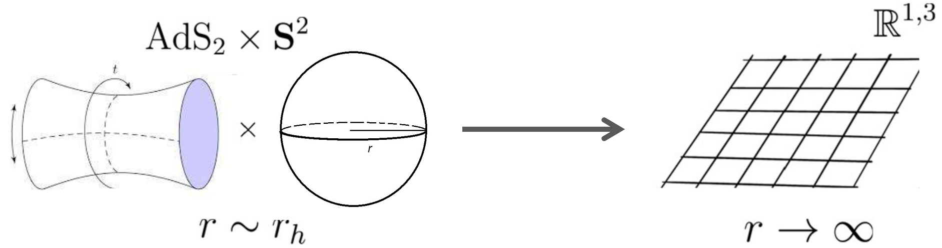

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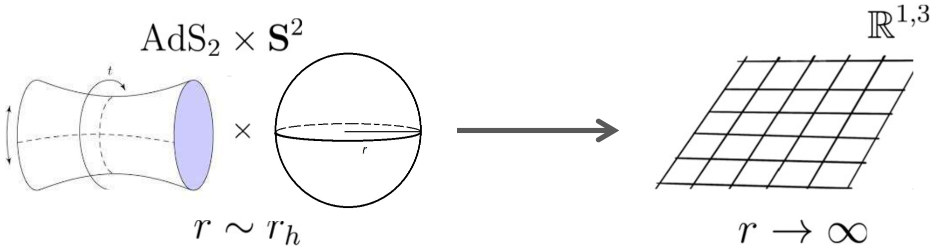

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4d N=2 Theories: BPS BHs

- The new central quantity is the generalized prepotential [Ooguri, Vafa, Strominger '04]

Higher derivative BPS effects included!

$$F(X,W^2) = \sum_{g=0}^{\infty} F_g(X^A) W^{2g} \qquad \text{with} \quad F_g(Y^A) = (-1)^g\, 2^{-6g} \mathcal{F}_g(Y^A)$$

- In terms of this the attractor equations read as usual [Behrndt et al '98]

$$ip^A = Y^A - \bar{Y}^A \qquad iq_A = F_A(Y,\Upsilon) - \bar{F}_A(\bar{Y},\bar{\Upsilon}) \qquad \text{with} \quad \Upsilon = -64$$

- The quantum-corrected entropy formula can also be determined to be

$$\mathcal{S}_{\rm BH} = \pi \left[|\mathcal{Z}|^2 + 4 \text{Im}\, \left(\Upsilon \partial_\Upsilon F(Y,\Upsilon) \right) \right]$$

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The Large Volume Regime

- Up to now we have kept things general (i.e. model-independent)
- To answer our original question, we henceforth focus on the large radius singularity
- There, the generalized prepotential reads as

$$F(Y,\Upsilon) = \frac{D_{abc}Y^aY^bY^c}{Y^0} + d_a\frac{Y^a}{Y^0}\Upsilon + G(Y^0,\Upsilon) + \mathcal{O}\left(e^{2\pi i z^a}\right)$$

$$D_{abc} = -\frac{1}{6}\mathcal{K}_{abc}$$

$$d_a = -\frac{1}{24}\frac{1}{64}c_{2,a}$$

- The (universal) leading quantum correction (due to constant maps) is given by

$$G(Y^0,\Upsilon) = -\frac{i}{2(2\pi)^3}\chi_E(X_3)\,(Y^0)^2\sum_{g=0,2,3,\ldots}c^3_{g-1}\,\alpha^{2g} + \ldots$$

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Asymptotic growth

Expansion parameter

$$\text{with}\quad c^3_{g-1} = (-1)^{g-1} 2(2g-1)\frac{\zeta(2g)\zeta(3-2g)}{(2\pi)^{2g}}\,, \qquad \alpha^2 = -\frac{1}{64}\frac{\Upsilon}{(Y^0)^2}$$

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Higher derivative corrections

- The attractor solutions simplify considerably. For instance, the entropy yields

$$\mathcal{S}_{\rm BH}=\pi\left[|\mathcal{Z}|^2-2id_a\left(\frac{Y^a}{Y^0}\Upsilon-\frac{\bar{Y}^a}{\bar{Y}^0}\bar{\Upsilon}\right)-2i\left(\Upsilon\frac{\partial G(Y^0,\Upsilon)}{\partial\Upsilon}-\bar{\Upsilon}\frac{\partial\bar{G}(\bar{Y}^0,\bar{\Upsilon})}{\partial\bar{\Upsilon}}\right)\right]$$

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Outline

I. Review: 4d N=2 BPS Black Holes

II. Gluing Entropies Across Dimensions

  i. The D0-D2-D4 System

  ii. Perturbative Corrections and Non-Local Resummation

  iii. Leading Non-Perturbative Effects

III. The Fate of Other BPS Systems

  i. The D2-D6 System

  ii. A closer look @ non-perturbative effects

IV. Summary and Outlook

Part II

Gluing Entropies Across Dimensions

The D0-D2-D4 BH System

- Consider BPS BHs with no D6-brane charge
- The two-derivative attractor solution is well known. We thus impose

$$W^2 \to 0\,, \quad F(X^A, W^2) \to \mathcal{F}(X^A)$$

- The solution reads [Shmakova '96]

$$CX^a = \frac{1}{6} CX^0 D^{ab} q_b + \frac{i}{2} p^a \qquad (CX^0)^2 = \frac{1}{4}\frac{D_{abc}p^a p^b p^c}{\hat{q}_0} \equiv (x^0)^2$$

- From here one may easily determine both the radius and the entropy of the BH system

$$\frac{r_h^2}{G_4} = |Z(q_A, p^B)|^2 = -\frac{D_{abc}p^a p^b p^c}{CX^0} = 2\sqrt{\frac{1}{6}|\hat{q}_0|\mathcal{K}_{abc}p^a p^b p^c}$$

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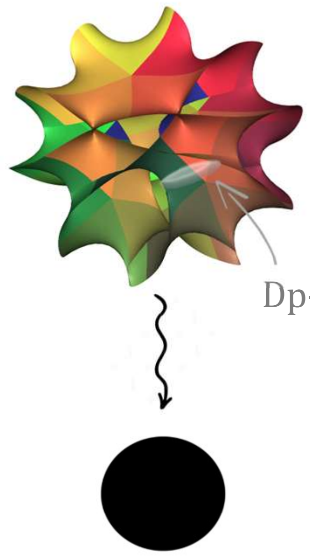

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The D0-D2-D4 BH System

- We assumed large vol approximation $\rightsquigarrow$ need to ensure that the solution is consistent!
- Due to monotonicity of BPS flow, we only have to worry about the horizon locus [Ferrara '95-'97]
- Compute stabilized volumes:

$$t^a_{\mathrm{h}} = \mathrm{Im}\left(\frac{CX^a}{CX^0}\right)\Big|_{\mathrm{hor}} = p^a\sqrt{\frac{6|\hat{q}_0|}{\mathcal{K}_{abc}p^a p^b p^c}}$$

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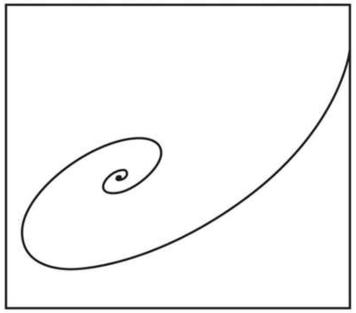

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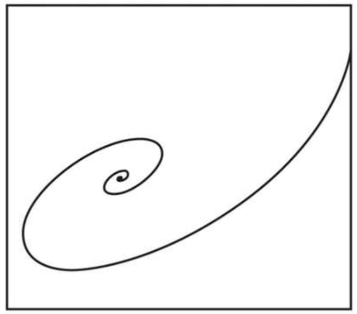

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We do not specify x_0

Including Perturbative Quantum Corrections

- Taking now the generalized prepotential with the leading quant. corrections yields

$$(Y^0)^2 = \frac{\frac{1}{4}D_{abc}p^a p^b p^c - d_a p^a \Upsilon}{\hat{q}_0 + i(G_0 - \bar{G}_0)}\,, \qquad \text{with} \;\; G_0 \equiv \frac{\partial G(Y^0,\Upsilon)}{\partial Y^0}$$

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$$Y^a = \frac{1}{6}Y^0 D^{ab} q_b + \frac{i}{2}p^a$$

Same as before!

- Notice that in order to recover the previous classical solution we need to impose

$$|\hat{q}_0| \gg p^a \gg 1\,, \qquad |\hat{q}_0| \gg |i(G_0 - \bar{G}_0)|$$

- One can thus find an iterative solution of the form

$$(Y^0)^2 = (y^0)^2\left(1 + \frac{i(G_0(y^0,\Upsilon) - \bar{G}_0(\bar{y}^0,\bar{\Upsilon}))}{|\hat{q}_0|} + \dots\right) \qquad \text{with} \;\; (y^0)^2 = (x^0)^2\left(1 - 4d_a p^a \Upsilon / D_{bce}p^b p^c p^e\right)$$

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Including Perturbative Quantum Corrections

- The corrected black hole entropy and radius read as [Lopes-Cardoso, Wit, Mohaupt '99]

$$|\mathcal{Z}|^2 = 2\sqrt{\frac{1}{6}|\hat{q}_0|\mathcal{K}_{abc}p^a p^b p^c} + \dots$$

$$\mathcal{S}_{\mathrm{BH}} = 2\pi\sqrt{\frac{1}{6}|\hat{q}_0|\left(\mathcal{K}_{abc}p^a p^b p^c + c_{2,a}\,p^a\right)} - 2\pi i\left(G(y^0,\Upsilon) - \bar{G}(\bar{y}^0,\bar{\Upsilon})\right) + \dots$$

- We can understand the condition $|\hat{q}_0| \gg |i(G_0 - \bar{G}_0)|$ by evaluating the series

$$i\left(\bar{G}_0 - G_0\right) = -\frac{\chi_E(X_3)}{(2\pi)^3}\,Y^0 \sum_{g=0,2,3,\dots}(2-2g)\,c^3_{g-1}\,\alpha^{2g} + \dots$$

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Including Perturbative Quantum Corrections

- The corrected black hole entropy and radius read as [Lopes-Cardoso, Wit, Mohaupt '99]

$$|\mathcal{Z}|^2 = 2\sqrt{\frac{1}{6}|\hat{q}_0|\mathcal{K}_{abc}p^a p^b p^c} + \ldots$$

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Now we require large values for x_0

The Transition Regime

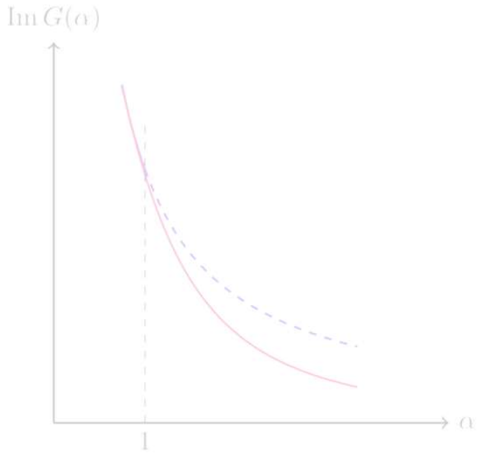
- Including the leading quant. corrections yields sensible answers for certain hierarchies
- The latter are controlled by a series expansion that is asymptotic (for $\alpha \ll 1$)

$$G(Y^0, \Upsilon) \sim -\frac{i}{2(2\pi)^3} \chi_E(X_3)\,(Y^0)^2 \sum_{k=0}^{\infty} c^3_{k-1}\,\alpha^{2k} \qquad \text{with} \quad c^3_{k-1} \sim (2k-3)!$$

- The optimal truncation can be determined to behave as $N_\star \sim \frac{1}{2}\left(1 + \frac{4\pi^2}{\alpha}\right)$
- Thus, the series is invalidated for $\alpha \gtrsim \mathcal{O}(1)$ $\rightsquigarrow$ Interpretation?
- The attractor eqs actually tell us the physical meaning of

$$|\alpha| = \frac{1}{8}\frac{|\Upsilon|^{1/2}}{|X^0|e^{\mathcal{K}/2}|\mathcal{Z}|} = \frac{\sqrt{8\mathcal{V}_{\rm h}}}{|\mathcal{Z}|} = \frac{r_5}{r_h}$$

since $M_{\rm D0} = \sqrt{8\pi}|X^0|e^{\mathcal{K}/2}/\kappa_4\,,\, r_h = |\mathcal{Z}|\kappa_4/\sqrt{8\pi}$



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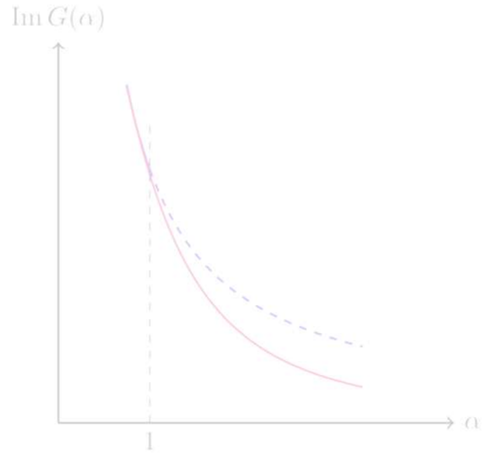

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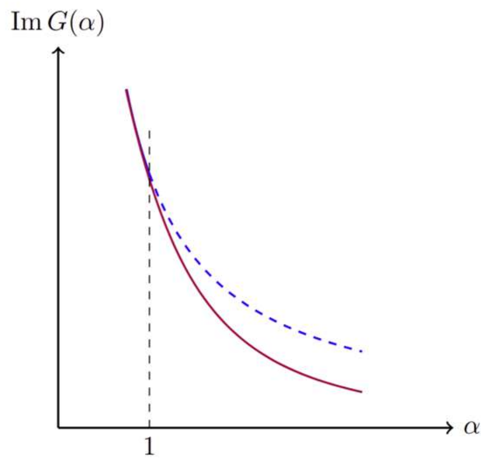

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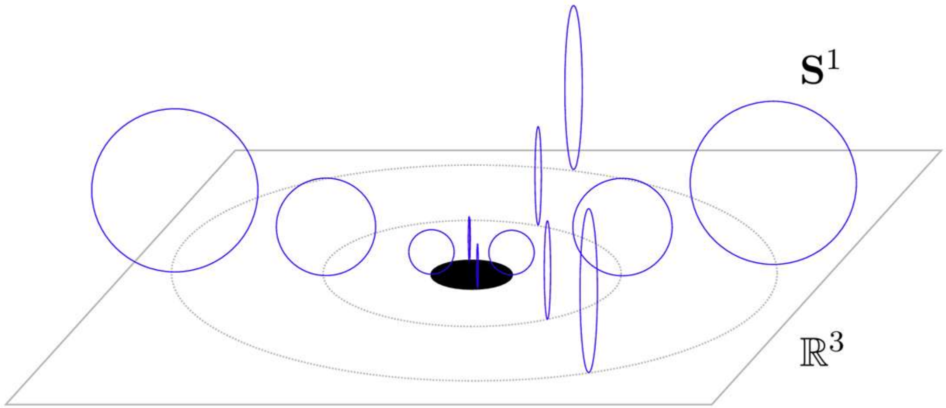
- The EFT fails to capture the relevant BH observable (i.e. entropy) when $r_h \lesssim r_{\rm KK}$



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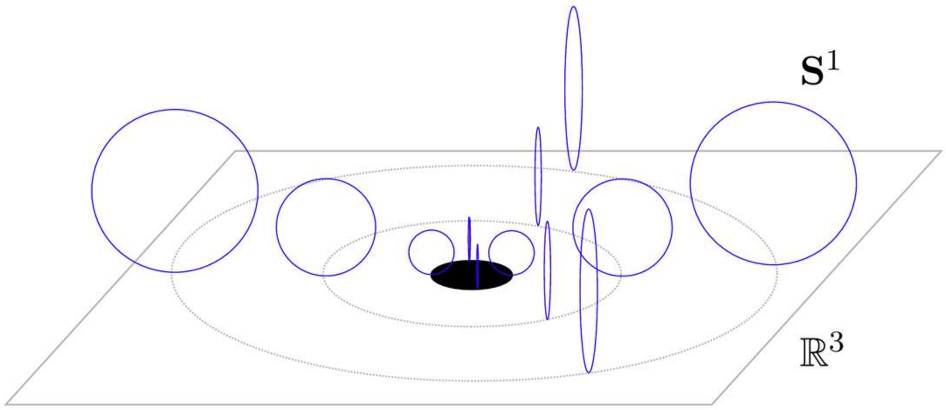

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Non-Local Resolution & EFT Transition

- Key observation: use the Gopakumar-Vafa representation of the topological free energy

[Gopakumar, Vafa '98]

$$G(Y^0,\Upsilon)=\frac{i}{2(2\pi)^3}\,\chi_E(X_3)\,(Y^0)^2\,\frac{\alpha^2}{4}\sum_{n\in\mathbb{Z}}\int_{0+}^{\infty}\frac{\mathrm{d}s}{s}\,\frac{1}{\sinh^2(\pi n\alpha s)}\,e^{-4\pi^2 n^2 is}\;=\;G^{(p)}(\alpha)\,+\,G^{(np)}(\alpha)$$

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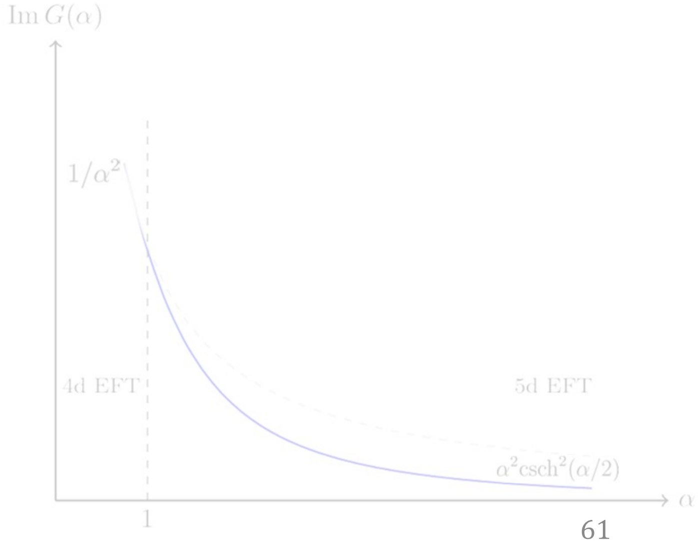
$$G^{(p)}(Y^0,\Upsilon)=\frac{i}{2(2\pi)^3}\,\chi_E(X_3)\,(Y^0)^2\,\alpha^2\sum_{n=1}^{\infty}n\,\mathrm{Li}_1\left(e^{-\alpha n}\right)$$



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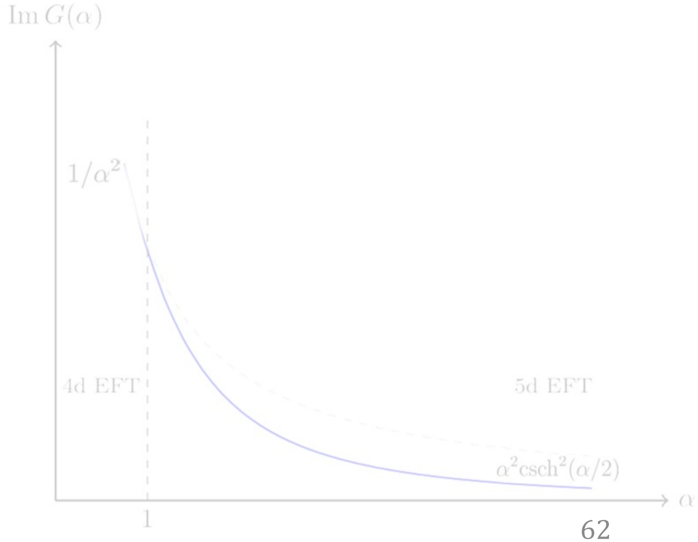

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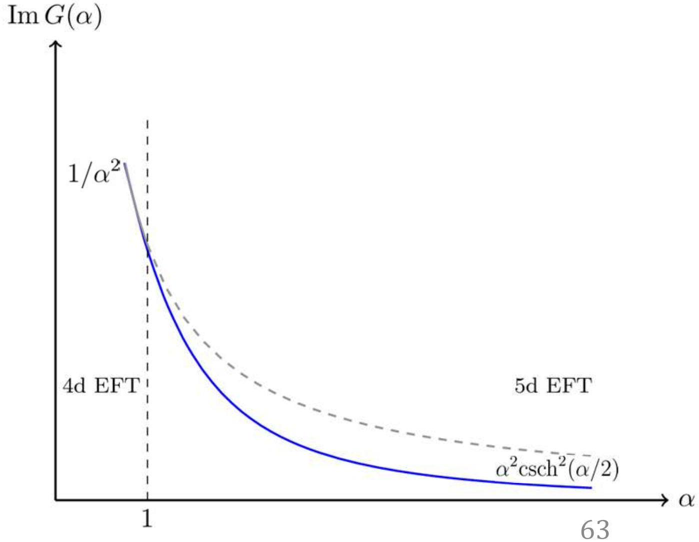

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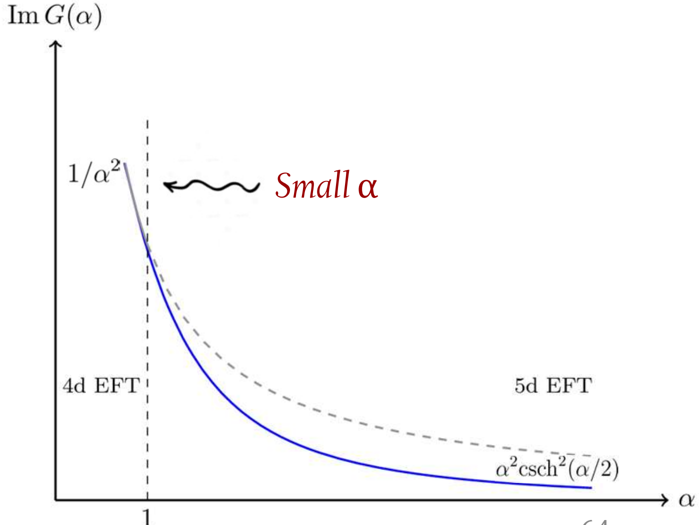

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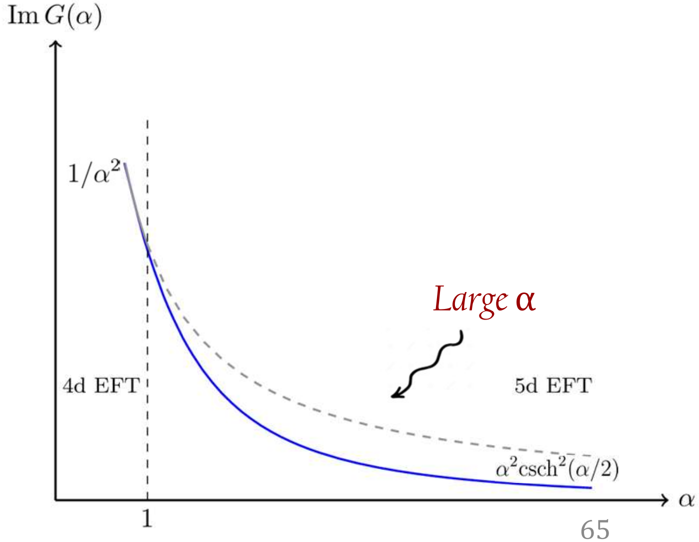

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- The BPS quantum extropy would then read as

$$\boxed{\begin{aligned}\mathcal{S}_{\rm BH} \;=\; & 2\pi\sqrt{\frac{1}{6}|\hat{q}_0|\left(\mathcal{K}_{abc}p^a p^b p^c + c_{2,\,a}p^a\right)}\left(1 - \frac{\chi_E(X_3)\,Y^0\,\alpha^2}{(2\pi)^3|\hat{q}_0|}\sum_{n=1}^{\infty} n^2\,\mathrm{Li}_0\left(e^{-\alpha n}\right)\right)^{-1/2}\\ & + \frac{\chi_E(X_3)}{4\pi^2}(Y^0)^2\alpha^2\left(\sum_{n=1}^{\infty} n\,\mathrm{Li}_1\left(e^{-\alpha n}\right) + (Y^0)^{-1}\sum_{n=1}^{\infty} n^2\,\mathrm{Li}_0\left(e^{-\alpha n}\right)\right)\end{aligned}}$$

Explicit Gluing with 5d Black Strings

- What are we getting in the 5d limit ($r_5 \gg r_h$)?

- The 4d BH lifts to a 5d black string wrapped on M-theory circle

$$\mathcal{S}_{\rm BH} \xrightarrow{\alpha\to\infty} 2\pi\sqrt{\frac{1}{6}|\hat{q}_0|\left(\mathcal{K}_{abc}p^a p^b p^c + c_{2,a}\, p^a\right)}$$

- What we obtained is nothing but the IR regulated infinite black string entropy!

- This matches perfectly the microscopic counting result [Maldacena, Strominger, Witten '97, Vafa '97]

$$\mathcal{S}_{\rm micro} = 2\pi\sqrt{\frac{|\hat{q}_0| c_L}{6}} \qquad c_L = \mathcal{K}_{abc}p^a p^b p^c + c_{2,a}\, p^a$$

- Remarkably, it includes the QG correction due to the R^2 [Sen '05, Kraus, Larsen '05, Castro et al '07]

- Notice that the minimal BH entropy arises when cubic and linear pieces compete!!

$$S_{\rm BH} \gtrsim \left(\frac{M_{\rm Pl}}{\Lambda_{\rm QG}}\right)^2 \qquad \text{with} \quad \Lambda_{\rm QG} \simeq M_{\rm Pl,5}$$

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Including Non-Perturbative Effects

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- Come back at Schwinger integral

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- One should be careful when evaluating the integral for positive/negative charged states

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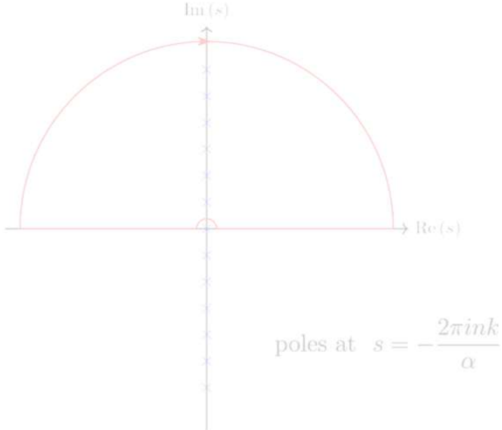
$$\mathcal{I}_{n<0}(\alpha) = \frac{\alpha^2}{4}\sum_{n\geq 1}\int_{0^-}^{-\infty}\frac{\mathrm{d}s}{s}\,\frac{e^{2\pi i n s}}{\sinh^2\left(\frac{\alpha s}{2}\right)}$$



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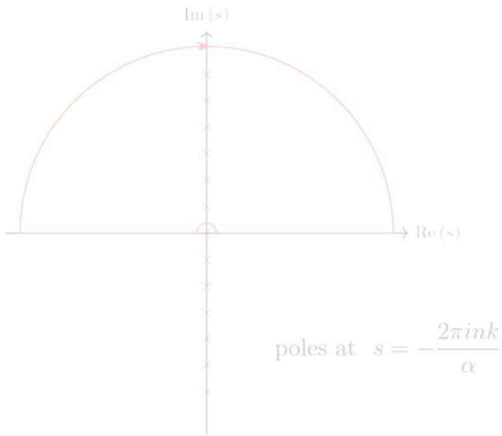

Including Non-Perturbative Effects

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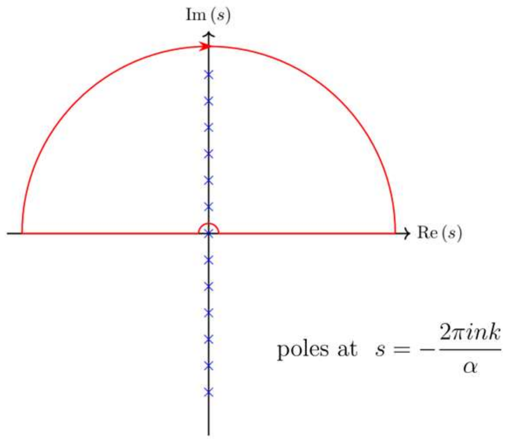

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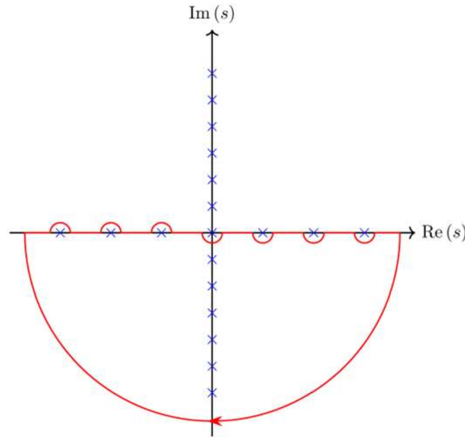

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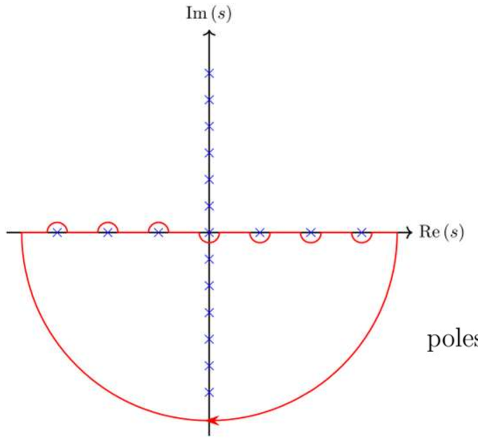

poles at $s = k\,,\&\ s = \frac{2\pi in}{\alpha}$

[see also Hattab, Palti '24]

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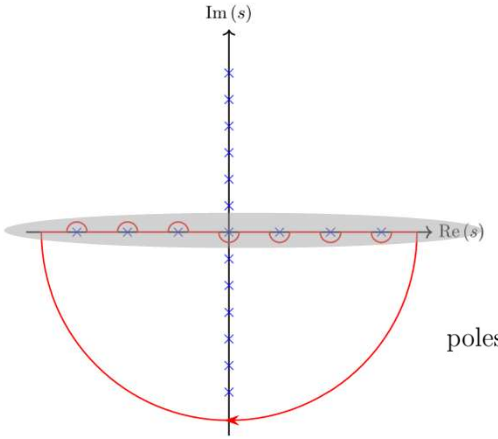

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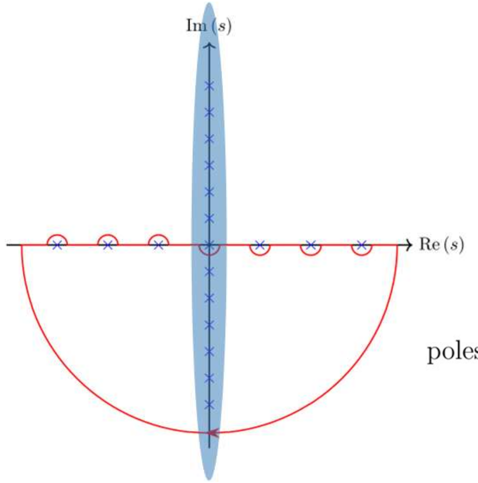
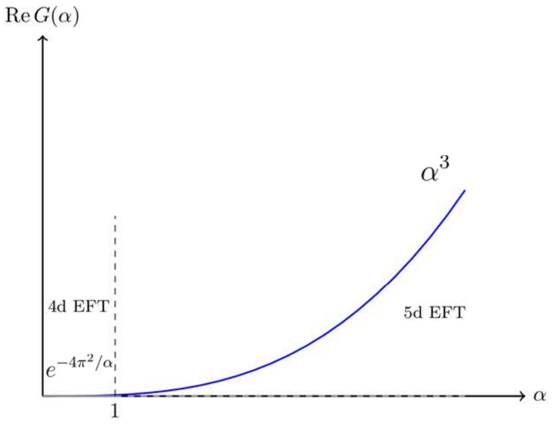


poles at $s = k\,,\; \&\; s = \frac{2\pi i n}{\alpha}$

Including Non-Perturbative Effects

- The non-perturbative correction is now easily determined

$$\mathcal{I}^{(np)}(\alpha) = -2\pi i\alpha \sum_{n,k=1}^{\infty} \frac{n}{k}\, e^{-\frac{4\pi^2 kn}{\alpha}} \left(1 + \frac{\alpha}{4\pi^2 kn}\right)$$
$$= -2\pi i\alpha \sum_{n=1}^{\infty} \left(n\,\mathrm{Li}_1\left(e^{-\frac{4\pi^2 n}{\alpha}}\right) + \frac{\alpha}{4\pi^2}\,\mathrm{Li}_2\left(e^{-\frac{4\pi^2 n}{\alpha}}\right)\right)$$



- Notice the problematic growth for $\alpha \gg 1$

- Crucially, this has a different complex phase, and it does not enter the att. eqs nor BH obvs!

$$(Y^0)^2 = \frac{\frac{1}{4}D_{abc}p^ap^bp^c - d_ap^a\Upsilon}{\hat{q}_0 + i(G_0 - \bar{G}_0)}$$

Attractor eq.

$$|\mathcal{Z}|^2 = -\frac{D_{abc}p^ap^bp^c - 2d_ap^a\Upsilon}{Y^0} + iY^0\left(G_0 - \bar{G}_0\right)$$
$$\mathcal{S}_{\mathrm{BH}} = -4\pi Y^0\hat{q}_0 - i\pi\left(3Y^0G_0 + 2\Upsilon G_\Upsilon - \mathrm{h.c.}\right)$$

BH entropy and radius

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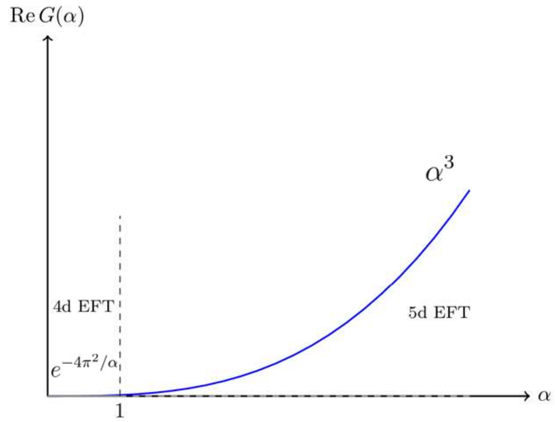

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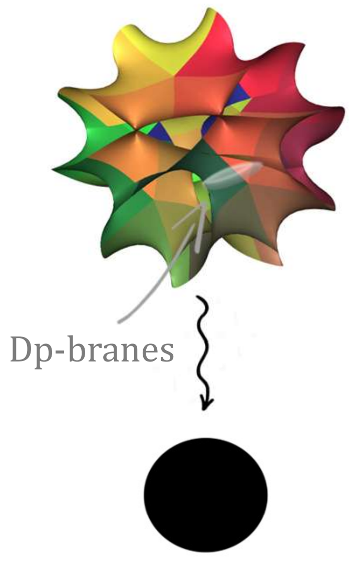
Outline

Part III

The Fate of Other BPS Black Hole Systems

The (Classical) D2-D6 BH System



- We would now like to study other BPS solutions which include D6-brane charge
- At 2-derivatives the problem is hard: we must deal with a quadratic alg. system
- We focus on a particularly simple system, i.e. the D2-D6 BPS black hole
- Having no D4 charge implies

$$CX^0 = \mathrm{Re}\, CX^0 + i\,\frac{p^0}{2} \qquad CX^a = \bar{C}\bar{X}^a = \mathrm{Re}\, CX^a$$

- The attractor equations read

$$D_{abc}\,(CX^b)(CX^c) = -\frac{q_a}{3p^0}|CX^0|^2$$

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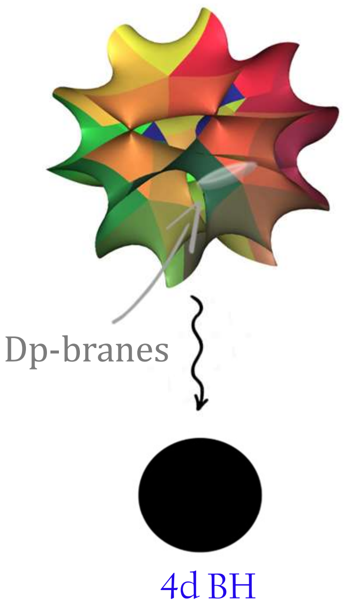

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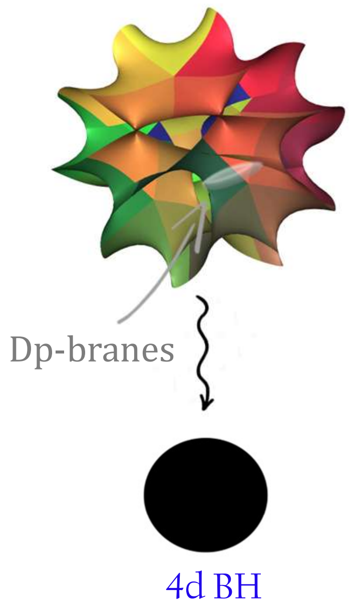

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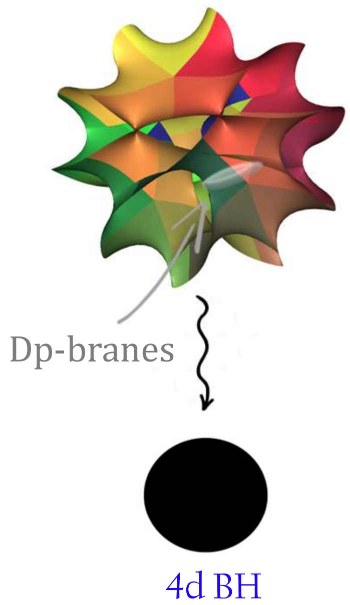

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$$q_0 \propto CX^0 = 0$$

The (Classical) D2-D6 BH System

- Defining the variables

$$x^a = \mathrm{Re}\, CX^a \sqrt{\frac{3}{|CX^0|^2}} \Longrightarrow D_{abc} x^b x^c = -\frac{q_a}{p^0}$$

- One can easily write the physical properties of the BHs such as the central charge

$$|Z|^2 = -\frac{4}{3}(q_a CX^a)$$

- ...as well as the relevant volumes (implying the hierarchy $q_a \gg p^0$)

$$\mathcal{V}_{\mathrm{h}} = \frac{1}{8}\frac{|Z|^2}{|CX^0|^2} = \frac{2}{3}\frac{(-q_a CX^a)}{(p^0)^2} = \frac{D_{abc} CX^a CX^b CX^c}{i\,(CX^0)^3} \qquad t^a_{\mathrm{h}} = \mathrm{Im}\left(\frac{CX^a}{CX^0}\right)\Big|_{\mathrm{hor}} = -2\,\frac{CX^a}{p^0} = -\frac{1}{2}\frac{p^0 CX^a}{|CX^0|^2}$$

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- The formal series of corrections is now alternating

$$G(Y^0,\Upsilon) = \frac{i}{2(2\pi)^3}\,\chi_E(X_3)\,|Y^0|^2 \sum_{g=0,2,3,\ldots} (-1)^g c^3_{g-1}\,|\alpha|^{2g}$$

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G_0 is purely real!

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$$\frac{\partial G(Y^0,\Upsilon)}{\partial Y^0} = \frac{\chi_E(X_3)}{2(2\pi)^3}|Y^0| \sum_{g=0,2,3,\dots} (-1)^g(2-2g)c^3_{g-1}\,|\alpha|^{2g}$$

G_0 is purely real!

- The solution is not spoiled! The relevant BH quantities are given by

$$|\mathcal{Z}|^2 = -\frac{4}{3}Y^a\left(q_a - \frac{1}{12p^0}c_{2\,a}\right) + p^0\,\mathrm{Re}\,G_0$$

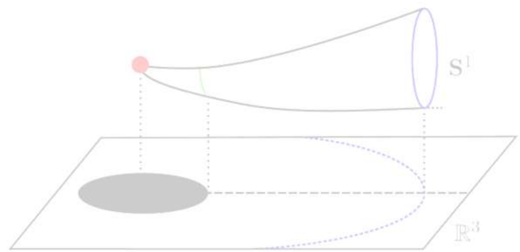
$$\mathcal{S}_{\mathrm{BH}} = -\frac{4}{3}\pi Y^a\left(q_a + \frac{1}{6p^0}c_{2\,a}\right) + \frac{\chi_E(X_3)(p^0)^2}{4(2\pi)^2}\sum_{g=0,2,3,\dots}(-1)^g(2-2g)c^3_{g-1}\,|\alpha|^{2g}$$

No Genuine 5d Regime

- Interestingly, $\alpha = i|\alpha|$ is upper bounded due to charge quantization

$$\alpha = -i\,|\alpha| \qquad |\alpha| = \frac{2}{p^0}$$

- From M-theory this is easily understood geometrically
- The BH can be understood as a 5d BH at the center of a Taub-NUT
- Still, one may explore the $r_h \gtrsim r_5$ regime



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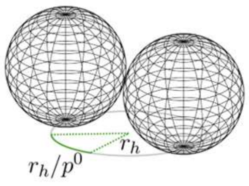

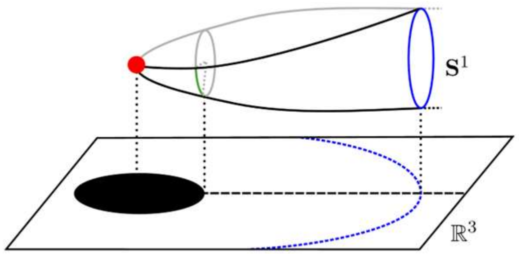

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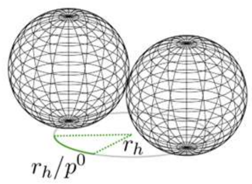

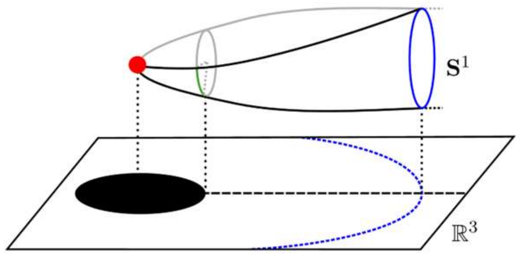

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Non Local and Non Perturbative Effects

- We resort again to the GV prescription, using that now $\alpha = i|\alpha|$

$$G(Y^0, \Upsilon) = -\frac{i}{2(2\pi)^3}\,\chi_E(X_3)\,|Y^0|^2\,\mathcal{I}(|\alpha|)$$

$$\mathcal{I}(|\alpha|) \;=\; \frac{|\alpha|^2}{4}\sum_{n\in\mathbb{Z}}\int_{0^+}^{\infty}\frac{\mathrm{d}s}{s}\,\frac{1}{\sin^2\left(\pi n|\alpha|s\right)}\,e^{-4\pi^2 n^2 i s}$$

- Now we can freely deform the contour of integration without picking poles!

- The resulting integral can be performed numerically

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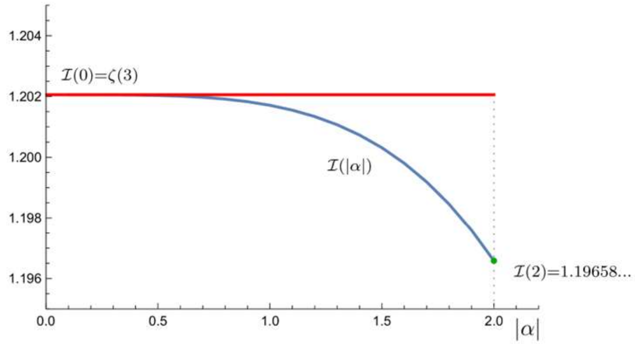

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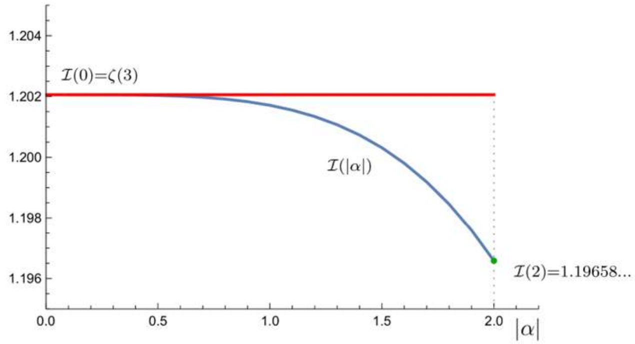

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Challenges with the Cauchy Formulation

- Question: Can we always use the simple Cauchy formula? [see also Hattab, Palti '24]

$$\mathcal{I}(\alpha) = \frac{\alpha^2}{4} \oint \frac{\mathrm{d}s}{s} \frac{1}{1-e^{-2\pi i s}} \frac{1}{\sinh^2\left(\frac{\alpha s}{2}\right)} \qquad \alpha = |\alpha| e^{i\theta_\alpha} \in \mathbb{C}$$

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$$s = \frac{2\pi n}{|\alpha|} \exp\left(i\pi/2 - i\theta_\alpha\right), \qquad n \in \mathbb{Z}$$

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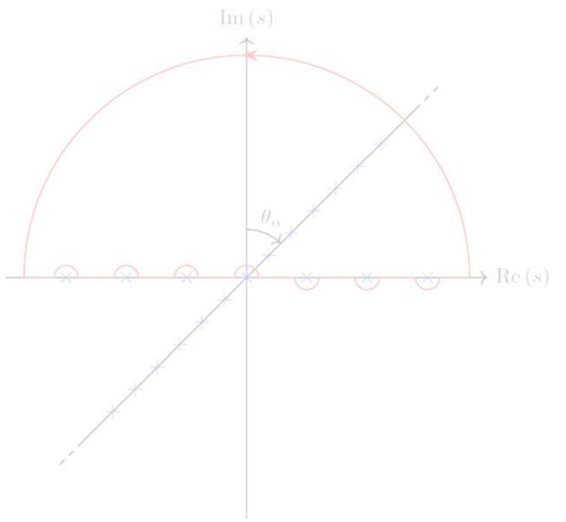

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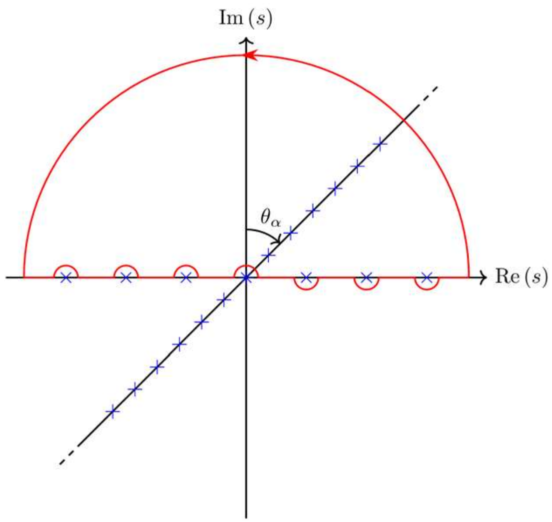

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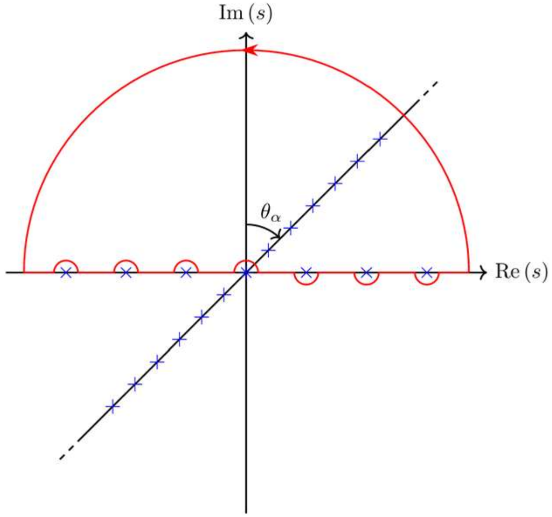

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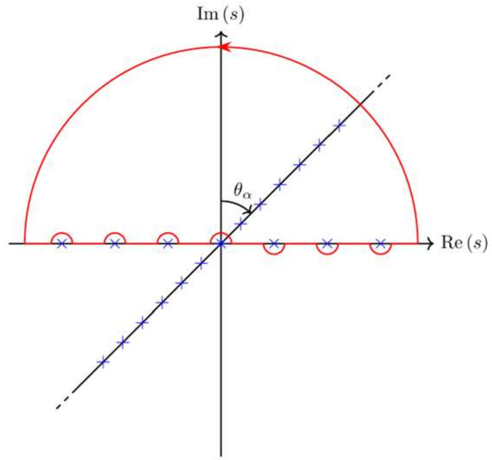

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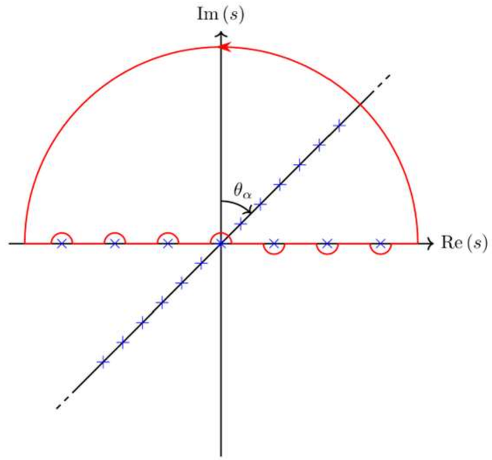

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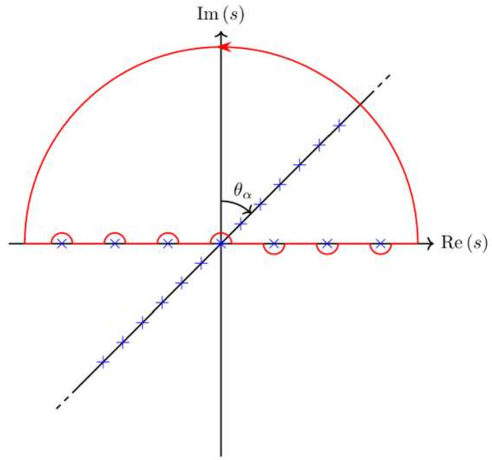

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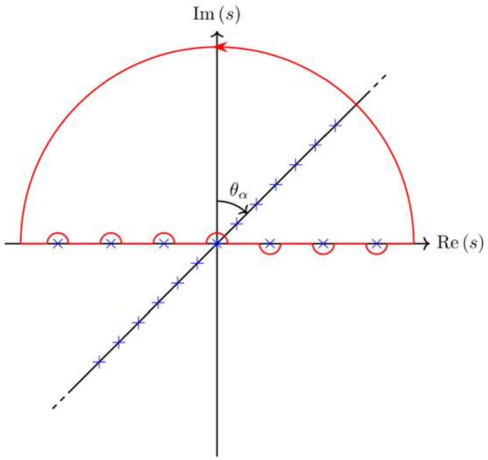

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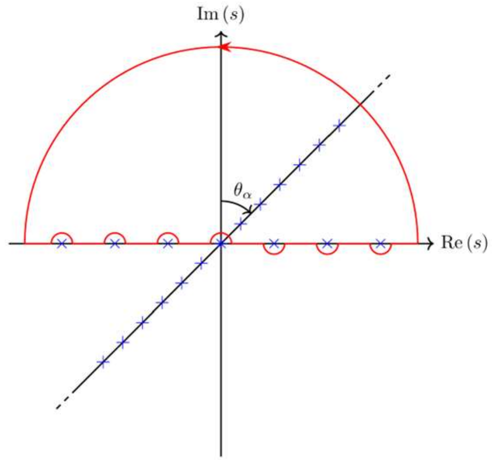

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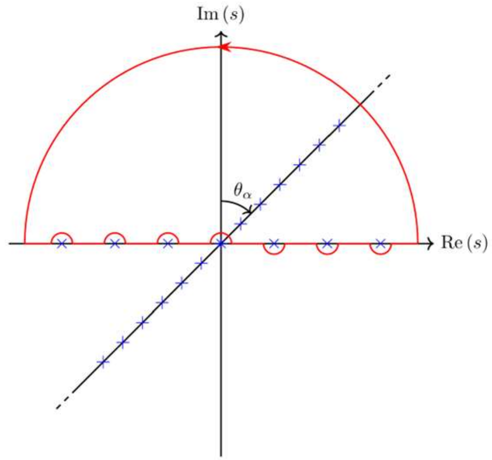

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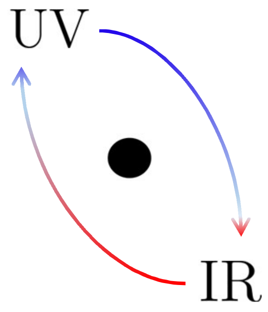
Part IV

Summary and Outlook

Summary & Outlook

- We have illustrated how extremal BHs can probe the multi-scale structure of gravity
- At curvatures/energies around M there is an EFT transition
  1. The EFT gives wrong/misleading predictions for BH observables
  2. This can be cured by resuming the quantum corrections



- At curvatures/energies around Λ_{QG} we reach the minimal BH entropy
- We illustrated this in 2 particular examples: D0-D2-D4 and D2-D6 systems
  1. Asymptotic series breaks down at dual M-theory circle scale
  2. Non-local effects allow to resum and dilute the corrections in the 5d regime
  3. Only the QG suppressed effects survive
  4. Non-perturbative phenomena do not spoil the analysis

Conclusions & Outlook

- There are many possible extensions of our work
  1. Going beyond large volume (e.g. include WS instantons)
  2. BHs probing the F-theory limit in elliptic CYs [WIP]
  3. BHs probing weakly coupled string phases [WIP]
  4. Small BHs [WIP]
- It is also important to understand the fate of non-pert. effects in the general case [WIP]
- One should also revisit the GV computation in $AdS_2 x S^2$ [WIP]
- Stay tuned!

KADANOFF CENTER
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ENRICO FERMI
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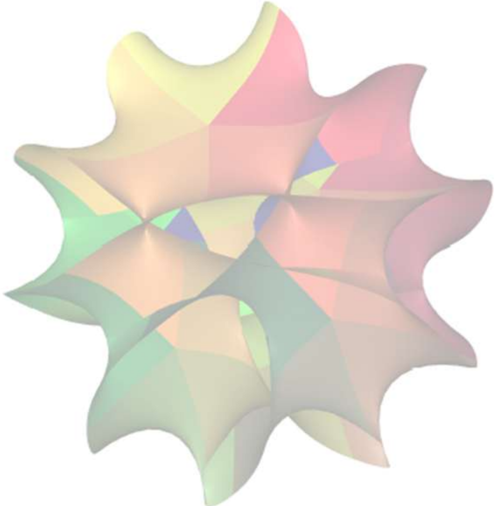
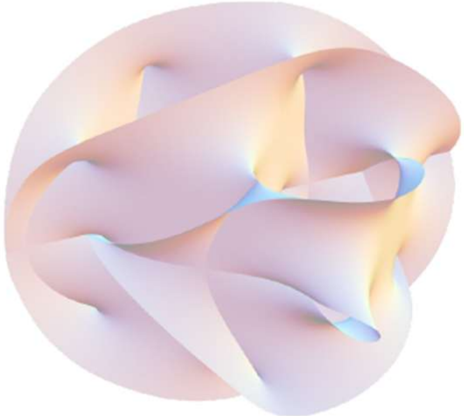
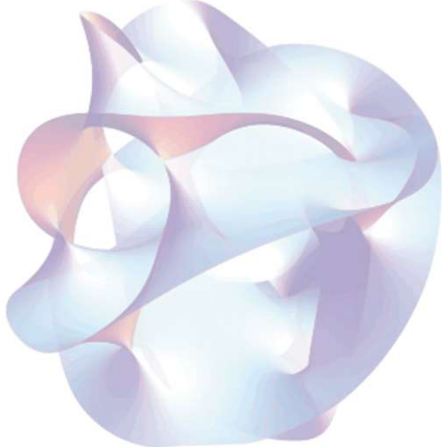
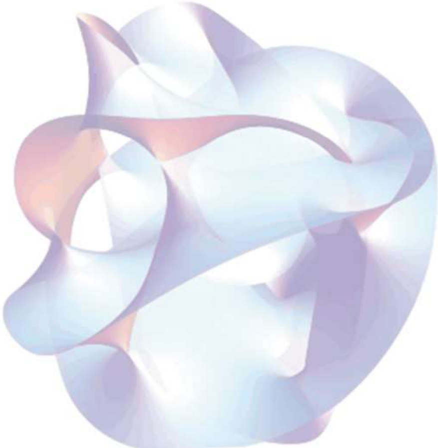
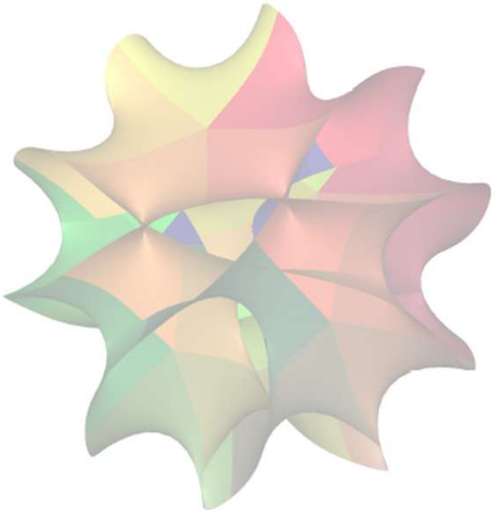
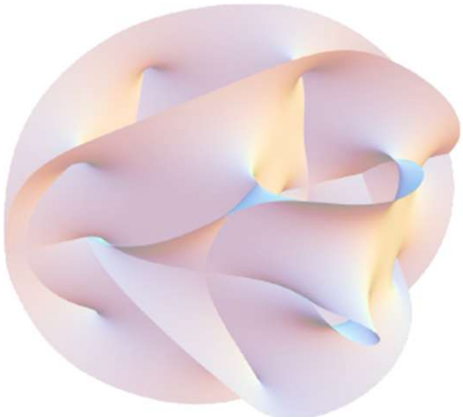
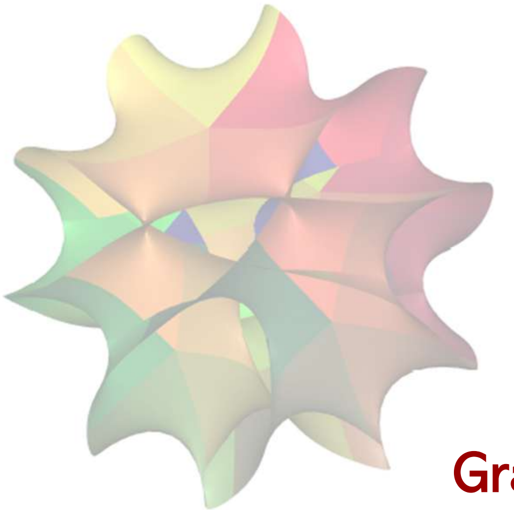
Thank you for your attention!

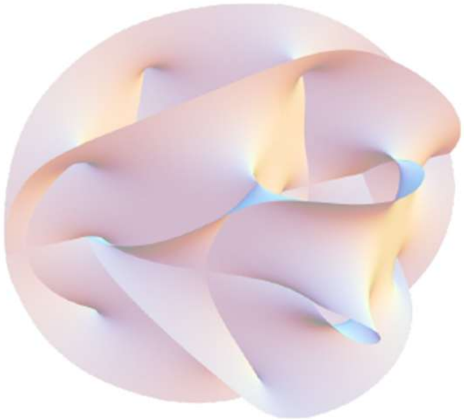
Contact: acastellano@uchicago.edu

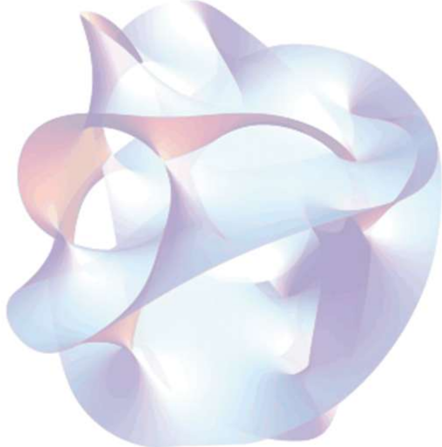




Back-up Slides

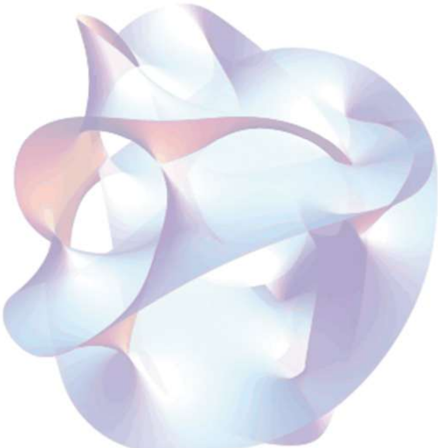




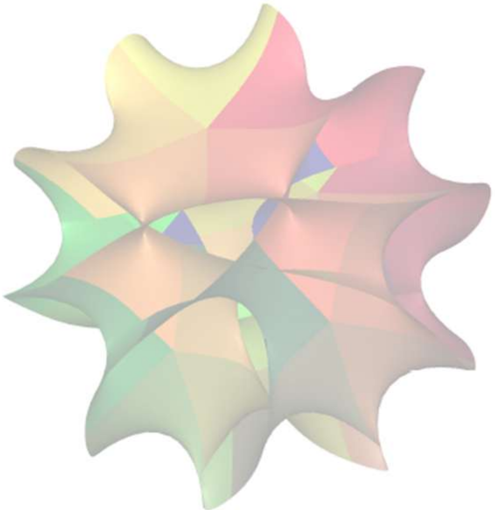


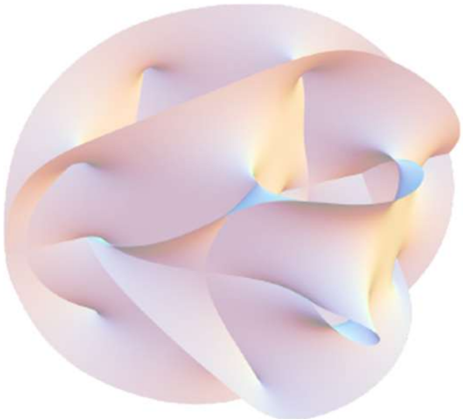




Gravity and the Species Scale



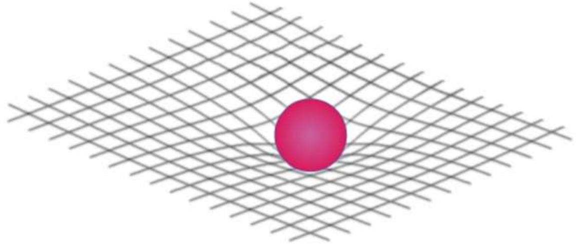




The Species Scale

- Gravity is non remormalizable

$$S_{\rm EH}\left[g_{\mu\nu}\right] = \frac{1}{2\kappa_d^2}\int d^dx\sqrt{-g}\left(\mathcal{R} - 2\Lambda_{\rm c.c.}\right)$$



- Recall that G_N is precisely the coupling constant

$$\mathcal{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R} + \Lambda_{\rm c.c.}\, g_{\mu\nu} = 8\pi G_N T_{\mu\nu} \qquad \text{with} \quad T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_{\rm matter}}{\delta g^{\mu\nu}}$$

- The most natural guess for $\Lambda_{\rm QG}$ is thus the energy scale associated to G_N

$$\Lambda_{\rm QG} := \kappa_d^{-\frac{1}{d-2}} = M_{\rm Pl;\, d}$$

- Hence, the EFT expansion for gravity should read as [e.g., Donoghue '94]

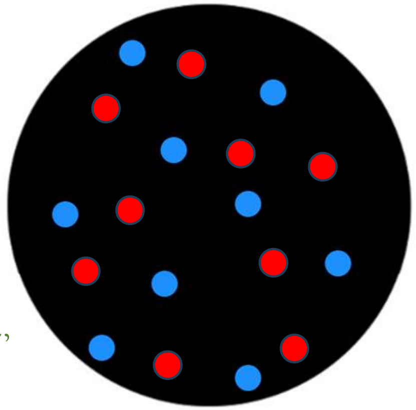
$$S_{\rm EFT}\left[g_{\mu\nu}\right] = \frac{1}{2\kappa_d^2}\int d^dx\sqrt{-g}\left(\mathcal{R} - 2\Lambda_{\rm c.c.} + \sum_{n\geq 2}\frac{\mathcal{O}_n(\mathcal{R})}{M_{\rm Pl;d}^{n-2}}\right)$$

Higher-curv. ops. are Planck suppressed!

The Species Scale

- Let's test this idea using well-motivated gravity principles [AC, Herráez, Ibáñez '21-24]
- Consider a spherical box in d spacetime dim
- How many field/metric configurations?

$N = \mathcal{O}(1)$



$$E = \mathfrak{c}_1 \, \mathsf{R}^{d-1} \, T^d$$
$$S = \mathfrak{c}_2 \, (\mathsf{R}\, T)^{d-1} \quad \Longrightarrow \quad S \lesssim A^{\frac{d-1}{d}} \ll \frac{A}{4}$$

[Bekenstein '72, Bousso '99]

$$R \gtrsim \left(\frac{E}{M_{\text{Pl;d}}^{d-2}} \right)^{\frac{1}{d-3}}$$

No collapse condition

- Well-established entropy bounds impose that minimal size is reached for $A = \mathcal{O}(1)$

The Species Scale

- But what if N is very large? [AC, Herráez, Ibáñez '21-24]
- Repeating the same exercise now yields

$$E = \mathfrak{c}_1\, N\, \mathsf{R}^{d-1}\, T^d \qquad \qquad S \le \mathfrak{c}_3 \left(N\, A^{d-1}\right)^{\frac{1}{d}}$$
$$S = \mathfrak{c}_2\, N\, (\mathsf{R}\, T)^{d-1} \quad \Longrightarrow \quad \le \frac{A}{4}$$

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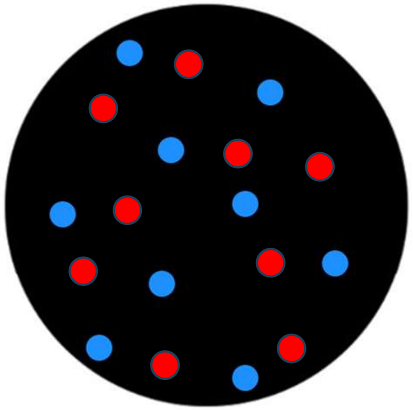

- To avoid violation of entropy bounds we need to impose $N \lesssim A$!

The Species Scale

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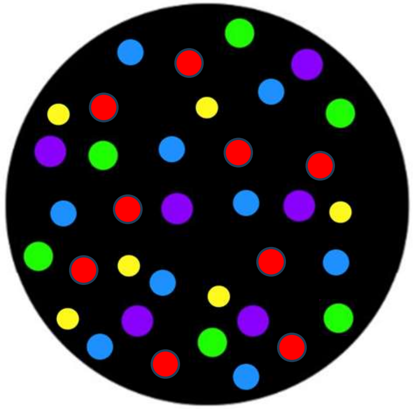
$N \gg 1$



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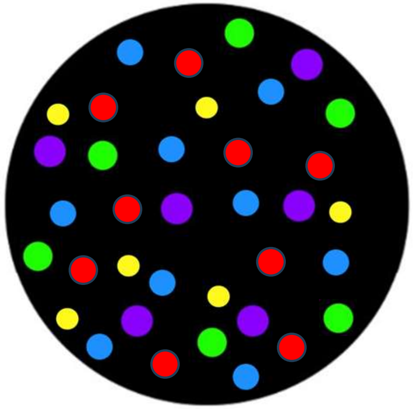

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$$R \gtrsim \left(\frac{E}{M_{\text{Pl};\text{d}}^{d-2}}\right)^{\frac{1}{d-3}}$$

- Minimal length in gravity is actually $\boxed{A \geq N \Longrightarrow \ell_{\text{min}} = \ell_{\text{sp}} := \ell_{\text{pl}} N^{\frac{1}{d-2}}}$

The Species Scale

- We thus define (asymptotically) the species scale as follows [Dvali, Redi '07; Dvali, Gómez '10]

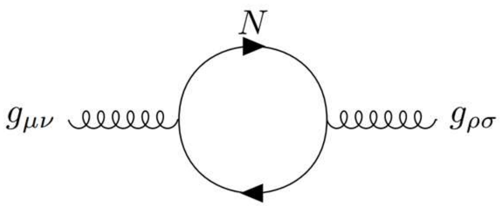
$$\Lambda_{\text{sp}} \approx \frac{M_{\text{Pl;d}}}{N^{\frac{1}{d-2}}} \lesssim M_{\text{Pl;d}}$$

- Notice that when N grows, Λ_{sp} and $M_{\text{Pl;d}}$ decouple!

- This is particularly interesting in light of Swampland conjectures

- There exist various arguments to arrive at the conclusion $\Lambda_{\text{QG}} = \Lambda_{\text{sp}}$ [Dvali '07]

    1. Perturbative (graviton series)
    2. Non-perturbative (Black holes)



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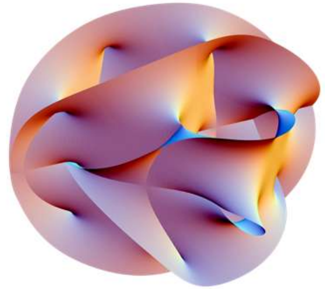
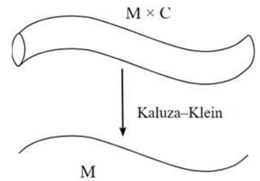
$$\boxed{\mathcal{L}_{{\rm EFT},d} \supset \sqrt{-g}\left[\frac{1}{2\kappa_d^2}\left(\mathcal{R} + \sum_{n>2}\frac{\mathcal{O}_n(\mathcal{R})}{\Lambda_{\rm sp}^{n-2}}\right)\right] + ({\rm matter})}$$

[v. d. Heisteeg, Vafa, Wiesner, Wu '22-23
AC, Herráez, Ibáñez '23]

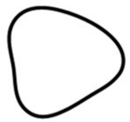
One Scale to Rule them All



- Actually, from string theory, the fact that $\Lambda_{\rm QG} \neq M_{\rm Pl;\,d}$ is not that surprising
- In fact, $M_{\rm Pl;d}$ typically depends on the starting theory & details of the compact.

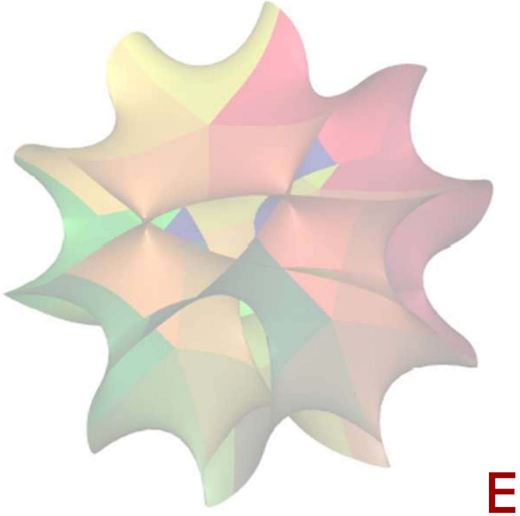


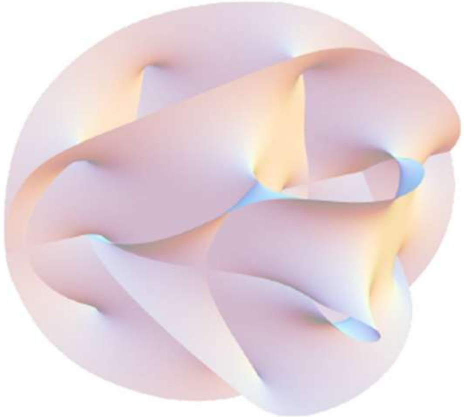
1. In decompact. limits one obtains $\Lambda_{\rm sp} \sim M_{\rm Pl;\,d+k}$
2. For weak coupling points we find $\Lambda_{\rm sp} \sim \sqrt{T_s}$

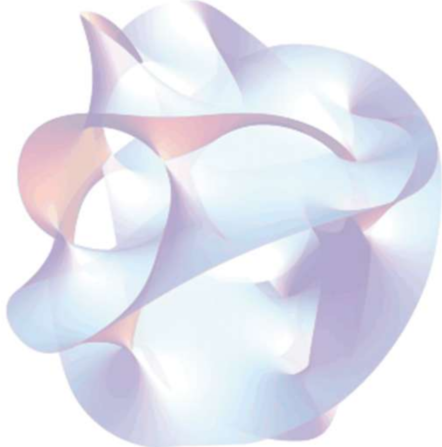


[AC, Herráez, Ibáñez '21-24]

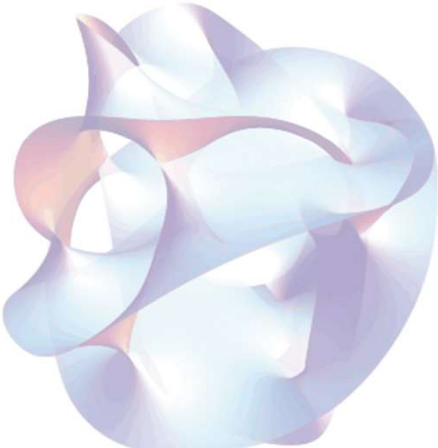
- Both limits are thus understood under the same concept within QG
- Moreover, it suggests that the appearance of light towers is **the** universal mechanism for quantum gravity 'phase transitions'

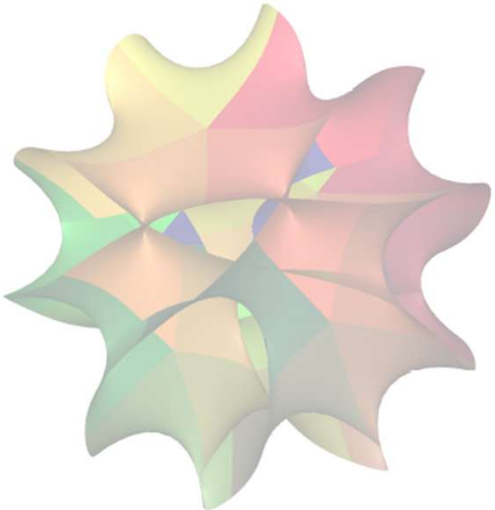


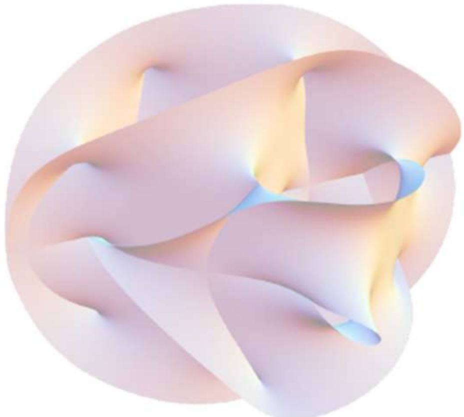




Entropy vs Entropy Index







What do we mean by Entropy?

- The previous formula arises by using Wald's formalism in a truncated theory [Lopes-Cardoso, Wit, Mohaupt '99]
- Essentially, one ignores D-term-like and hypermultiplet contributions
- Some of these were shown to give vanishing corrections [Lopes-Cardoso et al. '00, Murthy, Reys '13]
- It is believed that what we actually compute is a grav. index [Ooguri, Vafa, Strominger '04]

$$\mathcal{S}_{\rm BH} = \log \mathcal{Z}_{\rm index} - iq\phi \qquad \text{with} \quad \mathcal{Z}_{\rm index} = \mathrm{Tr}\Big[(-1)^F e^{iq\phi}\Big]_{\rm susy} = \sum_q (-1)^F \Omega(p,q)\, e^{iq\phi}$$

- In the large charge expansion one would have

$$\mathcal{Z}_{\rm index} \sim \mathcal{Z} \qquad \Longrightarrow \mathcal{S}_{\rm micro} = \log \Omega(p,q) \sim \mathcal{S}_{\rm BH}$$

[Zaffaroni '19]

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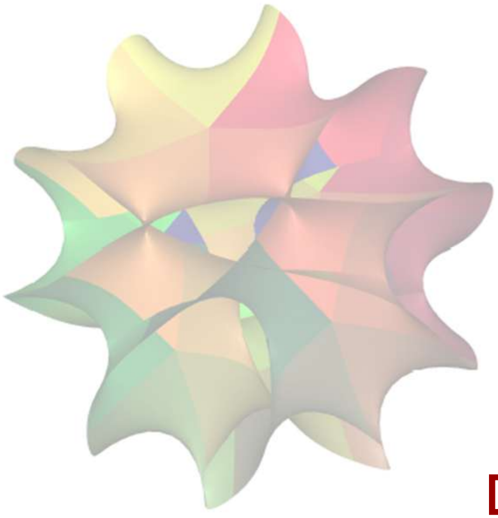
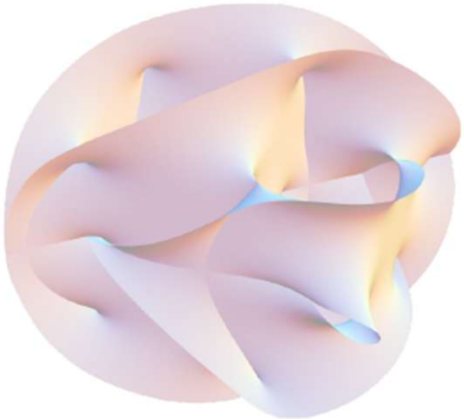
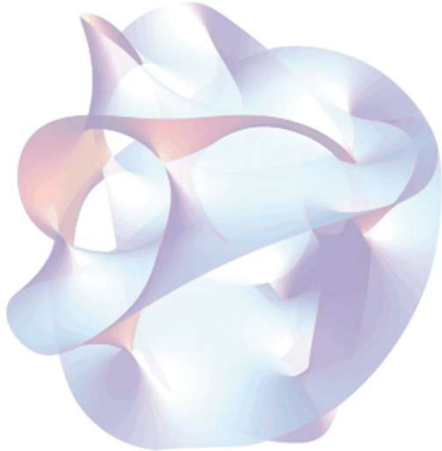
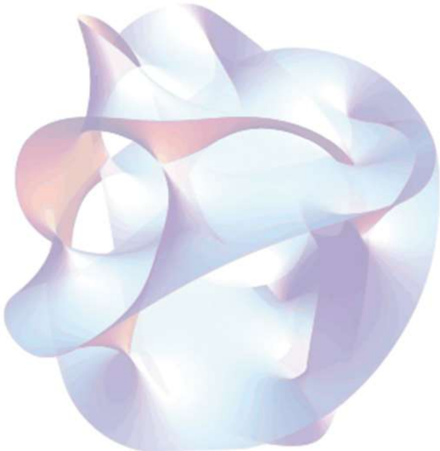
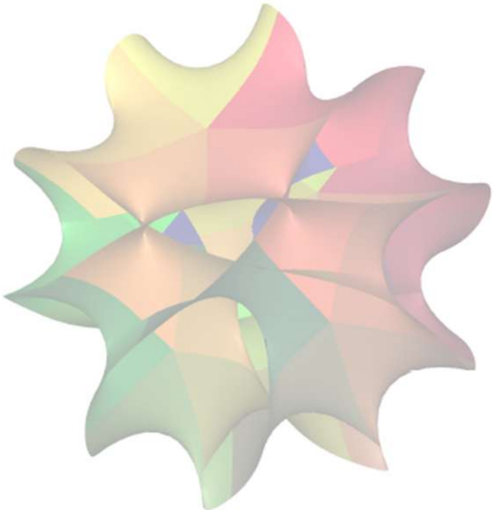


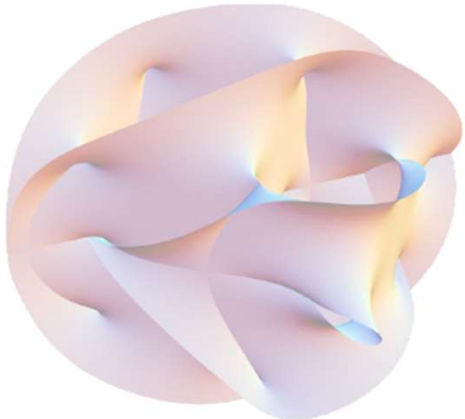




Details on Gopakumar-Vafa







Higher genus free energies

- Beyond two derivatives, there exist interesting higher-curvature BPS operators in 4d N=2

$$S_{\rm IIA}^{\rm 4d} \supset \int d^4x\,\sqrt{-g}\,\Big(\sum_{g\geq 1} \mathcal{F}_g(X^A)\,\mathcal{R}_+^2\,F_+^{2g-2}\Big) \;+\; \text{h.c.}$$

Scalars *Graviton & graviphoton (self-dual)*

- The Wilson 'coefficients' are computed by topological string theory [Antoniadis, Gava, Narain, Taylor '95]

$$\begin{aligned}\sum_{g\geq 0} \mathcal{F}_g\, F_+^{2g-2} &= -\frac{1}{4}\int_{0^+}^{i\infty} \frac{d\tau}{\tau}\frac{1}{\sin^2 \frac{\tau F_+ \bar{Z}}{2}} e^{-\tau m^2} \\ &= \frac{1}{4}\int_{0^+}^{\infty}\frac{d\tau}{\tau}\sum_{g\geq 0}\frac{2^{2g}(2g-1)}{(2g)!}(-1)^g B_{2g}\left(\frac{\tau F_+}{2}\right)^{2g-2} e^{-\tau Z} + \mathcal{O}\left(e^{-\frac{Z}{F_+}}\right)\end{aligned}$$

[Gopakumar, Vafa '98]

- Alternatively, one may use Gopakumar-Vafa prescription: integrating-out procedure

Higher genus free energies

- The latter approach makes manifest the UV behaviour [AC, Herráez, Ibáñez '23]

$$\mathcal{F}_g \propto \int_\varepsilon^\infty d\tau\, \tau^{2g-3} e^{-\tau Z} = Z^{2-2g}\Gamma(2g-2, \varepsilon Z) \qquad \text{with} \quad \varepsilon = \Lambda_{\rm UV}^{-2}$$

Central charge

- For $g \geq 2$ the loop integral converges, whereas for $g = 0, 1$ one needs to properly regularize!
- Let us briefly consider the case $g = 1$, corresponding to the $\mathcal{R}_+^2$ operator
- World-sheet computation: [Cecotti, Fendley, Intriligator, Vafa '93]

$$\mathcal{F}_1 = \frac{1}{2}\int \frac{d^2\tau}{\tau_2}\, \mathrm{tr}\left((-1)^F F_L\, F_R\, e^{2\pi i H_0}\, e^{-2\pi i \bar{H}_0}\right)$$

It is an index!

Higher genus free energies

- This can be integrated exactly [Bershadsky, Cecotti, Ooguri, Vafa '93]

$$\mathcal{F}_1 = \frac{1}{2}\left(3 + h^{1,1} - \frac{\chi_E(X_3)}{12}\right) K_{\rm ks} + \frac{1}{2}\log\det G_{i\bar{j}} + \log|f|^2$$

- For any infinite distance boundary one indeed finds [v.d. Heisteeg, Vafa, Wiesner, Wu '23]

$$\mathcal{F}_1 \sim \left(\frac{M_{\rm Pl;4}}{\Lambda_{\rm sp}}\right)^2$$

In agreement with expectations!

- E.g., for Enriques CY $\left(K3 \times \mathbf{T}^2\right)/\mathbb{Z}_2$ we find (@ large torus volume)

$$\mathcal{F}_1 = -6\log\left(T_2|\eta(T)|^4\right) + \text{const.} = 2\pi T_2 + \mathcal{O}\left(\log T_2\right) \sim \frac{M^2_{\rm Pl;4}}{T_{\rm NS5,\,str}}$$

Dual heterotic string

Higher genus free energies

- For $g \geq 2$ the situation is different (and more interesting)
- We find the same behaviour for all 3 diff. kinds of limits: decomp. To M/F-theory or emergent string limits [AC, Herráez, Ibáñez '23]
- For illustration purposes, we focus on the simplest one: the M-theory (large vol) limit
- The dominant contribution to $\mathcal{F}_{g>1}$ comes from D0-brane tower

$$m_n = 2\pi |n| \frac{m_s}{g_s} = |n|\, m_{\mathrm{D0}} \qquad \forall n \in \mathbb{Z}$$

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$$\begin{aligned}
\mathcal{F}_{g>1}^{\mathrm{D0}} &= \chi_E(X_3)\frac{(2g-1)\zeta(2g)}{(2\pi)^{2g}}\sum_{n\in\mathbb{Z}}{}' \int_0^\infty d\tau\, \tau^{2g-3} e^{-\tau\, n\, m_{\mathrm{D0}}} \\
&= \chi_E(X_3)\frac{(2g-1)\zeta(2g)}{(2\pi)^{2g}}\Gamma(2g-2) m_{\mathrm{D0}}^{2-2g}\sum_{n\in\mathbb{Z}}{}' \frac{1}{n^{2g-2}} \\
&= \chi_E(X_3)\frac{2(2g-1)\zeta(2g)\Gamma(2g-2)}{(2\pi)^{2g}}\frac{\zeta(2g-2)}{m_{\mathrm{D0}}^{2g-2}}
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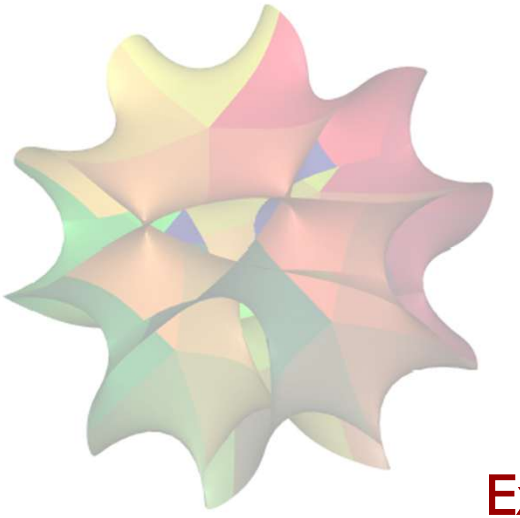
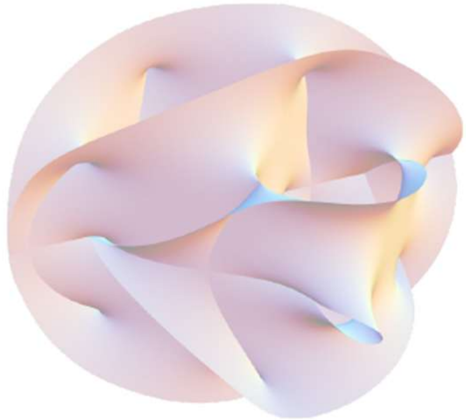
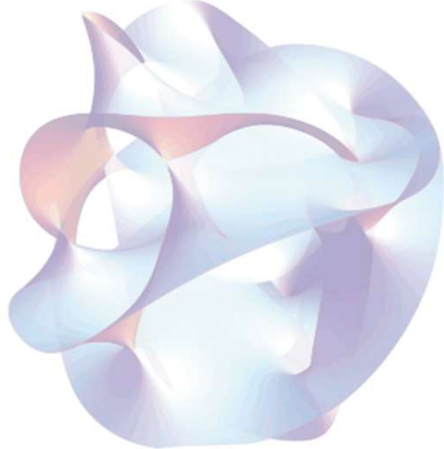

Extending Some Results